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Holomorphic mappings preserving Minkowski functionals

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ABSTRACT

We show that the equality $m_1(f(x)) = m_2(g(x))$ for x in a neighborhood of a point a remains valid for all x provided that f and g are open holomorphic maps, $f(a) = g(a) = 0$ and m_1, m_2 are Minkowski functionals of bounded balanced domains. Moreover, a polynomial relation between f and g is obtained.

As a consequence of our considerations we extend the main result of Berteloot and Patrizio (2000) [2] and we simplify its proof.

We also show how to apply our results to quasi-balanced domains.
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1. Introduction and statement of result

The paper is motivated by results obtained in [2]. The main result is as follows.

Theorem 1.1. *Let m_1 and m_2 be Minkowski functionals of bounded balanced domains in $\mathbb{C}^m$ and let U be a domain in $\mathbb{C}^k$, $k \geq m$. Let $f, g : U \to \mathbb{C}^m$ be holomorphic mappings such that $f(a) = g(a) = 0$ and f and g are open in a neighborhood of a for some $a \in U$. Let $q \in \mathbb{R}$. Assume additionally that $m_1(f(x)) = (m_2(g(x)))^q$ for x in some neighborhood $V \subset U$ of a.*

Then q is a positive rational number and

(1) *$m_1 \circ f(x) = (m_2 \circ g(x))^q$ for all $x \in U$,*
(2) *f and g are related in the following sense: there is a $p \in \mathbb{N}$ and there are homogeneous polynomials ξ_k of degree kq, $k = 1, \ldots, p$, (if $kq \notin \mathbb{N}$, then $\xi_k \equiv 0$) such that*

$$f(x)^p + f(x)^{p-1}\xi_1(g(x)) + \cdots + \xi_p(g(x)) = 0, \quad x \in U. \tag{1}$$

Let us explain the notation occurring above. First of all recall that a mapping f is said to be *open in a neighborhood of a* if there is a neighborhood of a such that the restriction of f to this neighborhood is open. For $z, w \in \mathbb{C}^n$ put $z \cdot w = (z_1 w_1, \ldots, z_n w_n)$; z^k, $k \in \mathbb{Z}$, is understood analogously (i.e. $z^k := z \cdot \ldots \cdot z$, $z^{-1} = (z_1^{-1}, \ldots, z_n^{-1})$).

Moreover, the unit disk in the complex plane is denoted by $\mathbb{D}$ and $\partial_s \Omega$ stands for the Shilov boundary of a bounded domain Ω in $\mathbb{C}^n$.

Theorem 1.1, interesting in its own, has some important applications. For example, it is the main tool which allows us to generalize and simplify the proof of the main theorem of [2]. The proof presented here is quite elementary and does not use advanced tools of pluripotential theory — the key point relies upon the investigation of the Shilov boundaries of bounded balanced domains.

The paper is organized as follows. We start with the proof of Theorem 1.1 (it is divided into few steps). Next we present some examples and applications. Moreover, we show how the results for circular domains may be easily extended to quasi-circular ones.

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2. Proof of the main theorem, remarks and examples

Proof of Theorem 1.1. Losing no generality we may assume that $a = 0$ and $m \geq 2$. Moreover, it is clear that $q \in \mathbb{Q}_{>0}$. Take $p_1, p_2 \in \mathbb{N}$ such that $q = \frac{p_1}{p_2}$.

Step $1'$ First we focus our attention on the case when $k = m$. It follows from Remmert's theorem (see [7]) that 0 is an isolated point of $g^{-1}(0)$ and $f^{-1}(0)$. Therefore, shrinking V if necessary we may assume that $f|_V$ is proper onto image. Moreover, there is a domain V' such that $0 \in V' \subset V$, $g|_{V'}$ is also proper onto image and $g^{-1}(0) \cap V' = \{0\}$. Put $\mathcal{V} = g(\{x \in V' : \det g'(x) = 0\})$ and fix $\delta > 0$ such that $\Omega_2 = \{x \in \mathbb{C}^m : m_2(x) < \delta\}$ and $\Omega_1 = \{x \in \mathbb{C}^m : m_1(x) < \delta^q\}$ are relatively compact in $g(V')$ and $f(V)$, respectively. Since $V' \cap g^{-1}(0) = \{0\}$, one can see that $g^{-1}(\Omega_2)$ is a domain.

Take $x_0 \in \partial_s\Omega_2 \setminus \mathcal{V}$ and let G_j, $j = 1, \ldots, p$, be local inverses to $g|_{V'}$ defined in a neighborhood of x_0, i.e. $g^{-1} = \{G_1, \ldots, G_p\}$. It follows from the invariance of the Shilov boundary under proper holomorphic mappings (see [5, Theorem 3]) that there is an index i (fixed from now on) such that $G_i(x_0) \in \partial_s g^{-1}(\Omega_2)$. Put $y_0 := f(G_i(x_0))$. Since $g^{-1}(\Omega_2) = f^{-1}(\Omega_1)$ we may apply the argument from [5] again to state that $y_0 \in \partial_s\Omega_1$.

We aim at showing that the map

$$t \mapsto \frac{f \circ G_i(tx_0)}{t^q}$$

(defined in a neighborhood of 1) is constant. Put $\psi_x(t) := \frac{f \circ G_i(tx)}{t^q}$, $t \in \mathbb{D}(1, r) := \{\lambda \in \mathbb{C} : |\lambda - 1| < r\}$, where r is sufficiently small. This would simply follow from the fact that ψ_{x_0} maps $\mathbb{D}(r, 1)$ into $\bar{\Omega}_1$ and $\psi(1) \in \partial_s\Omega_1$.

Assume the contrary, i.e. ψ_{x_0} is non-constant. Then there is $0 < r' < r$ such that $y_0 \notin \psi_{x_0}(\partial\mathbb{D}(1, r'))$. Using the uniform convergence argument one can easily see that there is an $\epsilon > 0$ and there is a neighborhood $U(x_0) \subset g(V') \setminus \mathcal{V}$ of x_0 such that ψ_x is well defined in a neighborhood of $\overline{\mathbb{D}(1, r')}$ (decrease r' if necessary) and $\operatorname{dist}(y_0, \psi_x(\partial\mathbb{D}(1, r'))) > \epsilon$ whenever $x \in U(x_0)$.

Let $V(x_0)$ be an open neighborhood of the point x_0 such that $V(x_0) \subset U(x_0)$ and $\operatorname{dist}(y_0, V(y_0)) < \frac{\epsilon}{2}$, where $V(y_0) = f(G_i(V(x_0)))$.

Since y_0 lies in the Shilov boundary of Ω_1, there is an $F \in \mathcal{O}(\Omega_1) \cap \mathcal{C}(\overline{\Omega_1})$ such that $\max\{|F(x)| : x \in V(y_0) \cap \overline{\Omega_1}\} > \max\{|F(x)| : x \in \overline{\Omega_1} \setminus V(y_0)\}$ (otherwise the Shilov boundary of Ω_1 would be contained in $\overline{\Omega_1} \setminus V(y_0)$). Choose $\tilde{y} \in V(y_0) \cap \overline{\Omega_1}$ at which the maximum on the left side is attained and note that taking $y' \in \Omega_1 \cap V(y_0)$ sufficiently close to $\tilde{y}$ we get the following inequality:

$$|F(y')| > \max\{|F(y)| : x \in \overline{\Omega_1} \setminus V(y_0)\}. \tag{2}$$

Let $x' \in V(x_0)$ be such that $y' = f(G_i(x'))$.

First, observe that $m_1(y') = m_1(f(G_i(x'))) = m_2(g(G_i(x')))^q = m_2(x')^q$, so $x' \in \Omega_2$. Note also that $m_1(\psi_{x'}(t)) = m_2(x')^q$, hence $\psi_{x'}(\overline{\mathbb{D}(1, r')}) \subset \Omega_1$. Moreover, $\psi_{x'}(1) = y'$ and $\psi_{x'}(\partial\mathbb{D}(1, r')) \cap V(y_0) = \varnothing$.

But a function $F \circ \psi_{x'}$ attains its maximum on $\partial\mathbb{D}(1, r')$. This contradicts (2).

Step $1''$ It is clear that $\mathcal{V} \subset \{x \in g(V') : \Phi(x) = 0\}$ for some holomorphic function Φ on $g(V')$, $\Phi \not\equiv 0$ (the function Φ may be given explicitly — for example one may take $\Phi(x) = \prod_{j=1}^{p} \det g'(G_j(x))$ where G_j are local inverses to g).

Define $\tilde{\Psi}(t, x, y) := \prod_{i,j}(f(G_i(x)) - t^{p_1} f(H_j(y)))$, $x, y \in g(V')$, $t \in \mathbb{D}$, where G_i, H_j are local inverses to G defined in a neighborhood of x and y, respectively. Put $\Psi(t, x) := \tilde{\Psi}(t, t^{p_2}x, x)$, $x \in g(V')$, $t \in \mathbb{D}$. It follows easily from *Step* $1'$ that for every $x \in \partial_s\Omega_2 \setminus \mathcal{V}$ the mapping $\Psi(\cdot, x)$ vanishes in a neighborhood of 1. Hence $\Psi(t, x) = 0$ for any $t \in \mathbb{D}$ and $x \in \partial_s\Omega_2 \setminus \mathcal{V}$. Therefore, for a fixed $t \in \mathbb{D}$ the mapping $\Phi \cdot \Psi(t, \cdot)$ vanishes on $\partial_s\Omega_2$, so by the properties of the Shilov boundary $\Phi \cdot \Psi \equiv 0$. Whence $\Psi \equiv 0$.

Fix $x' \in \Omega_2 \setminus \mathcal{V}$, $l \in \{1, \ldots, m\}$ and observe that there is an i such that $f_l(G_i(t^{p_2}x)) = t^{p_1} f_l(G_i(x))$ for t in a neighborhood of 1 and x in a neighborhood of x'. We aim at showing that

$$f_l(G_j(t^{p_2}x')) = t^{p_1} f_l(G_j(x')) \tag{3}$$

for $j = 1, \ldots, p$ and t sufficiently close to 1. To prove it put $y_i = G_i(x')$ and $y_j = G_j(x')$. Note that y_i and y_j may be joined by a path $\gamma : [0, 1] \to g^{-1}(\Omega_1) \setminus \mathcal{U}$, where $\mathcal{U} = g^{-1}(\mathcal{V})$. Put $\Gamma = g \circ \gamma$. A standard compactness argument allows us to find a partition of the interval $0 = t_0 < t_1 < \cdots < t_n = 1$ and open balls $(B_k)_{k=1}^N$ covering Γ^*, $B_k \subset\subset \Omega_2 \setminus \mathcal{V}$, such that $\Gamma([t_{k-1}, t_k]) \subset B_k$ and preimage $g^{-1}(B_k)$ has exactly p connected components, $k = 1, \ldots, N$.

There is a unique holomorphic mapping H_1 on B_1 such that $g \circ H_1 = \mathrm{id}$ and $H_1(\Gamma(t)) = \gamma(t)$ for $t \in [t_0, t_1]$. Note that $H_1 = G_i$, so by the identity principle $f_l(H_1(t^{p_2}x)) = t^{p_1} f_l(H_1(x))$ for $x \in B_1$ and t sufficiently close to 1. Similarly, there is a holomorphic mapping H_2 on B_2 such that $g \circ H_2 = \mathrm{id}$, $H_2(\Gamma(t)) = \gamma(t)$ for $t \in [t_1, t_2]$ and $H_1 = H_2$ on $B_1 \cap B_2$. Using the identity principle again we get the relation $f_l(H_2(t^{t_2}x)) = t^{p_1} f_l(H_2(x))$ for $x \in B_2$ and t sufficiently close to 1. Proceeding inductively one may construct a mapping H_N holomorphic on B_N such that $H_N = H_{N-1}$ on $B_{N-1} \cap B_N$, $g \circ H_N = \mathrm{id}$, and $G_N(x') = H_N(\Gamma(t_N)) = \gamma(1) = y_j$. Moreover $f_l(H_N(t^{p_2}x)) = t^{p_1} f_l(H_N(x))$, for $x \in B_N$ and t close to 1. Note that $H_N = G_j$ in a neighborhood of x' and this finishes the proof of (3).

Thus, we have shown that for any $x \in \Omega_2 \setminus \mathcal{V}$ the equality $f(G_j(t^{p_2}x)) = t^{p_1} f(G_j(x))$ remains valid for all $j = 1, \ldots, p$, and t sufficiently close to 1.

Step 1‴ For $(\lambda_{i,j})_{i=1,\ldots,p}^{j=1,\ldots,m} \subset \mathbb{C}$ let us consider the following system of equations:

$$\sum_{\sigma \in \Sigma_p} \prod_{k=1}^{p} (y_{j_k} - \lambda_{\sigma(k),j_k}) = 0, \quad \{j_1, \ldots, j_p\} \subset \{1, \ldots, m\},\ j_1 \leq \cdots \leq j_p, \tag{†}$$

with unknowns $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$, where Σ_p denotes the set of p-permutations. Note that for a given $(\lambda_{i,j})_{i=1,\ldots,p}^{j=1,\ldots,m}$ the system (†) has p solutions given by formulas $y = (\lambda_{i,1}, \ldots, \lambda_{i,m})$, $i = 1, \ldots, p$. To show it observe that $(\lambda_{i,1}, \ldots, \lambda_{i,m})$ solves (†), $i = 1, \ldots, p$. On the other hand any root of the equations in (†) with $j_1 = \cdots = j_p$ is of the form $(\lambda_{i_1,1}, \ldots, \lambda_{i_m,m})$. What remains to do is to show that it is of the form $(\lambda_{i,1}, \ldots, \lambda_{i,m})$. Since these computations are quite elementary and tedious, we omit them here.

Multiplying out we get mappings ξ_α^I, where $|\alpha| < p$ and $I = I(j_1, \ldots, j_p)$, such that

$$\sum_{\sigma \in \Sigma_p} \prod_{k=1}^{p} (y_{j_k} - \lambda_{\sigma(k),j_k}) = p! y_{j_1} \ldots y_{j_p} + \sum_{|\alpha|<p} \xi_\alpha^I(\lambda) y^\alpha.$$

Observe that ξ_α^I are homogeneous of order $p - |\alpha|$ and note that they are *quasi-symmetric* in the following sense:

$$\xi_\alpha^I(\lambda_{1,1}, \ldots, \lambda_{1,m}, \ldots, \lambda_{p,1}, \ldots, \lambda_{p,m}) = \xi_\alpha^I(\lambda_{\sigma(1),1}, \ldots, \lambda_{\sigma(1),m}, \ldots, \lambda_{\sigma(p),1}, \ldots, \lambda_{\sigma(p),m}) \quad \text{for any } \sigma \in \Sigma_p. \tag{4}$$

Therefore it is clear that $\zeta_\alpha^I := \xi_\alpha^I \circ f \circ g^{-1} := \xi_\alpha^I(f_1 \circ G_1, \ldots, f_m \circ G_1, \ldots, f_1 \circ G_p, \ldots, f_m \circ G_p)$ is a well defined holomorphic mapping on $g(V')$.

It follows from the above considerations that

$$\zeta_\alpha^I(t^{p_2}x) = t^{p_1(p-|\alpha|)} \zeta_\alpha^I(x) \quad \text{for all } x \in \Omega_2 \text{ and } t \in \mathbb{D}. \tag{5}$$

Now one may write down the Taylor expansion of ζ_α^I around 0 in order to verify that ζ_α^I are homogeneous polynomials of degree $q(p - |\alpha|)$ (obviously, if $q(p - |\alpha|) \notin \mathbb{N}$, then $\zeta_\alpha^I \equiv 0$).

Consider the following system of equations:

$$\Theta_I(x, y) := p! y_{j_1} \ldots y_{j_p} + \sum_{|\alpha|<p} \zeta_\alpha^I(x) y^\alpha = 0, \quad I = I(j_1, \ldots, j_p), \tag{6}$$

$\{j_1, \ldots, j_p\} \subset \{1, \ldots, m\}$, $1 \leq j_1 \leq \cdots \leq j_p \leq m$. First observe that

$$\Theta_I(t^{p_2}x, t^{p_1}y) = t^{pp_1} \Theta_I(x, y), \quad t \in \mathbb{C}. \tag{7}$$

Note also that for x lying sufficiently close to 0 the following property holds:

$$m_2(x)^q = m_1(y) \quad \text{for any root } y \text{ the system of equations } \Theta_I(x, \cdot) = 0. \tag{8}$$

To prove it take $x \in g(V')$. It follows from the definition of the mappings ζ_α that all roots of the Eq. (6) are given by formulas $y = f(x_i)$, where $g(x_i) = x$, $i = 1, \ldots, p$ (precisely $x_i = g^{-1}(x)$ if $x \notin \mathcal{V}$). The assumptions of the theorem imply that for such a solution y

$$m_1(y) = m_1(f(x_i)) = m_2(g(x_i))^q = m_2(x)^q,$$

which proves (8) for x sufficiently close to 0. Making use of (7) we find that the relation (8) holds for all x.

The equality $\Theta_I(g(x), f(x)) = 0$ holds in the neighborhood of 0, so by the identity principle $\Theta_I(g(x), f(x)) = 0$ for $x \in U$. This means that $f(x)$ is the root of the equations $\Theta_I(g(x), \cdot) = 0$ for any $x \in U$. It follows from (8) that $m_1 \circ f = (m_2 \circ g)^q$.

In order to prove the second assertion it suffices to repeat the above reasoning to the mappings ξ_α^I with $I = I(j, \ldots, j)$, $j = 1 \ldots, m$. To be more precise let us define

$$\tilde{\xi}_k(x) := \sum_{1 \leq i_1 < \cdots < i_k \leq p} x_{i_1} \ldots x_{i_k}, \quad x = (x_1, \ldots, x_p) \in \mathbb{C}^p, \tag{9}$$

$$\xi_k(\lambda) := (\tilde{\xi}_k(\lambda_1), \ldots, \tilde{\xi}_k(\lambda_m)), \quad \lambda = (\lambda_1, \ldots, \lambda_m) \in (\mathbb{C}^p)^m. \tag{10}$$

Put $\zeta_k := \xi_k \circ f \circ g^{-1}$ and

$$\Theta(x, y) := y^p - \zeta_1(x) y^{p-1} + \cdots + (-1)^p \zeta_p(x).$$

As before we prove that $\Theta(f, g) \equiv 0$.

Step 2 Now we shall show the theorem for $k > m$. It follows from Remmert's theorem that $\dim_0 f^{-1}(0) = k - m$. Using the basic properties of analytic sets one can find an m-dimensional vector space L in the Grassmannian $\mathbb{G}(m, k)$ such that 0 is an isolated point of $L \cap f^{-1}(0)$ and $L \cap g^{-1}(0)$. We lose no generality assuming that the space L is of the form $L = \{(x_1, \dots, x_m, \sum \alpha_j^{m+1} x_j, \dots, \sum \alpha_j^k x_j) : x_i \in \mathbb{C}\}$ for some $\alpha_j^l \in \mathbb{C}, j = 1, \dots, m, l = m+1, \dots, k$. Fix $r > 0$ such that the polydisc $(r\mathbb{D})^k$ is relatively compact in V. Let $\tilde{B}$ be an arbitrary infinite Blaschke product not vanishing on $\frac{1}{2}\mathbb{D}$ and define $B(\lambda) = \tilde{B}(\lambda r^{-1})$, $\lambda \in r\mathbb{D}$.

Put $\tilde{f} := (f, \psi^{p_1}) := (f, e^{p_1\varphi}(x_{m+1} - \sum \alpha_j^{m+1} x_j)^{p_1}, \dots, e^{p_1\varphi}(x_k - \sum \alpha_j^k x_j)^{p_1})$ and $\tilde{g} := (g, \psi^{p_2}) := (g, e^{p_2\varphi}(x_{m+1} - \sum \alpha_j^{m+1} x_j)^{p_2}, \dots, e^{p_2\varphi}(x_k - \sum \alpha_j^k x_j)^{p_2})$, where $\varphi(x_1, \dots, x_k) := \frac{1}{B(x_1)} + \dots + \frac{1}{B(x_k)}$. Observe that the mappings $\tilde{f}$ and $\tilde{g}$ are locally open in a neighborhood of 0 (as 0 is an isolated point of the fibers $\tilde{f}^{-1}(0)$ and $\tilde{g}^{-1}(0)$).

Put $|y| := |y_1| + \dots + |y_{k-m}|$, $y \in \mathbb{C}^{k-m}$, and

$$\nu_i(x, y) := \left(m_i(x)^{\frac{1}{p_i}} + |y|^{\frac{1}{p_i}}\right)^{p_i}, \quad (x, y) \in \mathbb{C}^k = \mathbb{C}^m \times \mathbb{C}^{k-m},\ i = 1, 2.$$

It is clear that the equality $\nu_1(\tilde{f}) = \nu_2(\tilde{g})^q$ holds in a neighborhood of 0. Applying the previous step we get a natural number p, homogeneous polynomials $\tilde{\zeta}_\alpha^I$ and corresponding maps $\tilde{\Theta}_I$ such that $\tilde{\Theta}_I(\tilde{g}, \tilde{f}) = 0$. Moreover, the system of equalities $\tilde{\Theta}_I(x, y) = 0$, $x, y \in \mathbb{C}^k$, implies that $\nu_2(x)^q = \nu_1(y)$.

Expanding we infer that

$$\tilde{\Theta}_I(\tilde{g}, \tilde{f}) = \tilde{\Theta}_I((g, \psi^{p_2}), (f, \psi^{p_1})) = \theta_I(g, f) + e^{\varphi} h_1 + \dots + e^{s\varphi} h_p$$

for some $s \in \mathbb{N}$, holomorphic maps h_i on U and a θ_I given by the formula $\theta_I(x, y) := \tilde{\Theta}_I((x, 0), (y, 0))$. Making use of the construction of φ we immediately state that $\theta_I(g, f) \equiv h_1 \equiv \dots \equiv h_p \equiv 0$. Therefore $\tilde{\Theta}_I((g, 0), (f, 0)) \equiv 0$. Whence $m_1(f(x)) = m_2(g(x))^q$ for all $x \in U$, as claimed.

The relation (1) may be shown analogously. □

Remark 2.1. The equality $m_1(f(x)) = m_2(x)^q$ in a neighborhood of 0, where f is a proper holomorphic map and m_1, m_2 are Minkowski functionals of pseudoconvex balanced bounded domains, is the key point of the proof of the main theorem in [2]. The authors investigated this equality with the help of advanced tools of the projective dynamic.

Note that in Theorem 1.1 the more general equality was considered (we did not even need the plurisubharmonicity) and the methods we were using were much simpler.

We would like to point out that the proof for the equality $m_1(f(x)) = m_2(x)^q$ is even much less complicated (in this case $p = 1$ and the other steps of the proof are not needed). More precisely, to prove the theorem in this case one may proceed in the following way: using the invariance of the Shilov boundary from [5] and basic properties of Shilov boundaries we get that $f(t^{p_2}x) = t^{p_1} f(x)$ for x in a neighborhood of 0 and t in a neighborhood 1, where $q = p_1/p_2$. From this equality we deduce that f extends to the whole $\mathbb{C}^m$ and $f(t^{p_2}x) = t^{p_1} f(x)$ for $x \in \mathbb{C}^n$, $t \in \mathbb{C}$. Thus f is a polynomial.

Note also, that the argument presented above does not require the theorem of Bell on proper holomorphic mappings between balanced domains.

Remark 2.2. The statement of Theorem 1.1 is clear if m_1 and m_2 are the Euclidean norms and f, g are arbitrary holomorphic mappings (as the Euclidean norm is $\mathbb{R}$-analytic). One may check that in this case $p = 1$.

Similarly, the statement of Theorem 1.1 is clear in the case when m_1, m_2 are operator norms (as the operator norm is $\mathbb{R}$-analytic except for an analytic set).

Remark 2.3. Note that in the case when $m = k$ and $q = 1$, the number p occurring in the statement of Theorem 1.1 is equal to the multiplicity of the mapping f (restricted to some neighborhood of 0). Note also that for $p = 1$ the mappings f and g are not necessary biholomorphic (but then $f = \zeta_1 g$ for a linear mapping ζ_1).

Assume that p occurring in Theorem 1.1 is equal to 2. Then we are able to solve the Eq. (1) and state that $f(x) = Q_1(g(x)) + \sqrt{Q_2(g(x))}$, where Q_1 is linear mapping, Q_2 is a homogeneous polynomial of degree 2, and the branch of the square is chosen so that $\sqrt{Q_1 \circ g}$ is holomorphic.

Generally, we cannot conjecture that Q_2 vanishes. Consider the following example: $m_i(x, y) = |(x, y)| = |x| + |y|$, $i = 1, 2$, $f(x, y) = \frac{1}{2}(x^2 + 2xy + y^2, x^2 - 2xy + y^2)$ and $g(x, y) = (x^2, y^2)$. Then obviously $|f(x, y)| = |g(x, y)|$, $Q_1(x, y) = 1/2(x + y, x + y)$ and $Q_2(x, y) = xy$.

Remark 2.4. The assumptions of the openness of the mappings f and g in a neighborhood of a are important. This is illustrated by the following example: $f(x, y) = (xy, x^2y)$, $g(x, y) = (xy, y)$ and $\|(x, y)\| = \max\{|x|, |y|\}$. Clearly $\|f(x, y)\| = \|g(x, y)\|$ if and only if $|x| \le 1$ or $y = 0$.

Note also that for any neighborhood U of 0 the images $f(U)$ and $g(U)$ are not analytic.

It is natural to ask whether the assumption of the openness may be weakened. We would like to point out that the answer to this question is obvious in the case $m = 2$ — it is sufficient to consider the Weierstrass polynomials of f and g. This reasoning however cannot be applied to $m \ge 3$.

3. Quasi-circular domains

Let $k_1, \ldots, k_n$ be natural numbers. A domain D of $\mathbb{C}^n$ is said to be $(k_1, \ldots, k_n)$-circular if

$$(\lambda^{k_1} x_1, \ldots, \lambda^{k_n} x_n) \in D \quad \text{whenever } \lambda \in \partial\mathbb{D},\ x = (x_1, \ldots, x_n) \in D. \tag{11}$$

If the formula (11) holds for any $\lambda \in \overline{\mathbb{D}}$, then D is said to be $(k_1, \ldots, k_n)$-balanced (or $(k_1, \ldots, k_n)$-complete circular).

A domain Ω is called to be *quasi-circular* (respectively *quasi-balanced*) if it is k-circular (resp. k-balanced) for some $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$.

For $k = (k_1, \ldots, k_n)$-balanced domain $D \subset \mathbb{C}^n$ one may define its *k-Minkowski functional* (a *quasi*-Minkowski functional) by the following formula:

$$\mu_{D,k}(x) := \inf\{\lambda > 0 : (\lambda^{-k_1} x_1, \ldots, \lambda^{-k_n} x_n) \in D\}, \tag{12}$$

$x = (x_1, \ldots, x_m) \in \mathbb{C}^n$. The introduced above function has similar properties as the standard Minkowski functional. Recall them for the convenience of the reader:

$$\mu_{D,k}(\alpha^{k_1} x_1, \ldots, \alpha^{k_n} x_n) = |\alpha| \mu_{D,k}(x), \quad x \in \mathbb{C}^n,\ \alpha \in \mathbb{C}, \tag{13}$$

$$D = \{x \in \mathbb{C}^n : \mu_{D,k}(x) < 1\}. \tag{14}$$

For $k = (k_1, \ldots, k_n) \in \mathbb{N}$ and $x \in \mathbb{C}^n$ denote $k \cdot x := (x_1^{k_1}, \ldots, x_n^{k_n})$.

Let D be a k-balanced domain and $\mu_{D,k}$ be the quasi-Minkowski functional associated with this domain. Put $\tilde{k}_j := \frac{k_1 \cdot \ldots \cdot k_n}{k_j}$, $\tilde{k} := (\tilde{k}_1, \ldots, \tilde{k}_n)$ and define $m(x) := \mu_{D,k}(\tilde{k}^{-1} \cdot x)^{k_1 \cdots k_n}$. One may check that m is radial. In particular, m is the Minkowski functional of a bounded balanced domain and it satisfies the property $m(\tilde{k} \cdot x) = \mu_{D,k}(x)^{k_1 \cdots k_n}$, $x \in \mathbb{C}^n$. On the other hand $\tilde{k} \cdot f$ is open provided that f is an open holomorphic mapping.

This simple observation leads us to the following.

Corollary 3.1. *Let μ_1, μ_2 be quasi-Minkowski functionals of quasi-balanced domains. Let $f, g : U \to \mathbb{C}^m$ be a holomorphic mapping such that $f(a) = g(a) = 0$, for some $a \in U \subset \mathbb{C}^k$, $k \geq m$. Assume that $q \in \mathbb{R}$. If $\mu_1(f(x)) = (\mu_2(g(x)))^q$ in a neighborhood $V \subset U$ of a and the restrictions $f|_{V'}, g|_{V'}$ are open, then $\mu_1 \circ f(x) = (\mu_2 \circ g(x))^q$ for all $x \in U$ and $q \in \mathbb{Q}_{>0}$.*

One can try to derive a counterpart of the second assertion of Theorem 1.1 in the case of quasi-Minkowski functionals. Since the possible formula is a little complicated and self-evident, we omit it here.

For more information on quasi-circular domains we refer the reader to [6].

4. Some applications

It is well known by Bell's result (see [1]) that any proper mapping f between complete circular domains such that f is non-degenerate (i.e. $f^{-1}(0) = \{0\}$) is a polynomial. So we may expand f in a series $f = \sum_{j=p}^{q} Q_j$, $p \leq q$, where Q_j are homogeneous of degree j. Let us introduce the following notation: $\rho(f) := Q_p$, $\varrho(f) := Q_q$.

The following was essentially proved in [2].

Proposition 4.1. *Let $D, \Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ be pseudoconvex balanced domains. Let $f_i : D \to \Omega_i$ be proper holomorphic mappings such that $f_i^{-1}(0) = \{0\}$, $i = 1, 2$. Assume that there are $m, M > 0$ such that $m\|f_2(x)\|^q \leq \|f_1(x)\| \leq M\|f_2(x)\|^q$, $x \in D$. Then $\mu_1(f_1(x)) = \mu_2(f_2(x))^q$, $x \in \mathbb{C}^n$. In particular, $\mu_1(\varrho(f_1)(x)) = \mu_2(\varrho(f_2)(x))^q$ and $\mu_1(\rho(f_1)(x)) = \mu_2(\rho(f_2)(x))^q$ for $x \in \mathbb{C}^n$, where μ_1 and μ_2 are Minkowski functionals of Ω_1, Ω_2, respectively.*

Thus, if f_1 is a homogeneous polynomial, then f_2 is homogeneous, as well.

Proof. It is well known that $g_{\Omega_1}(0, f_1(x)) = q g_{\Omega_2}(0, f_2(x))$. Therefore $\mu_1(f_1(x)) = \mu_2(f_2(x))^q$ for $x \in \Omega$. Applying Corollary 3.1 we state that $\mu_1(f_1(x)) = \mu_2(f_2(x))^q$ for $x \in \mathbb{C}^n$.

Considering the values of the equations $t^{-n_1}\mu_1(f_1(tx)) = t^{-n_1}\mu_2(f_2(tx))^q$ and $t^{n_2}\mu_1(f_1(x/t)) = t^{n_2}\mu_2(f_2(x/t))^q$ at $t = 0$ we easily get the second part of the assertion. $\square$

Remark 4.2. Suppose that D is a k-circular domain and consider the mapping $\pi : \mathbb{C}^n \ni z \mapsto k.z \in \mathbb{C}^n$. Then $\tilde{D} := \pi^{-1}(D)$ is a balanced domain and $\pi : \tilde{D} \to D$ is proper.

Let G be a complete circular domain. A simple argument together with Bell's theorem shows that any non-degenerate proper holomorphic mapping $f : D \to G$ is a polynomial and that it may be written as $f = \sum_{j \geq p} f_j$, where each term f_j is a k-homogeneous polynomial of order j (i.e. $f_j(t^{k_1} x_1, \ldots, t^{k_n} x_n) = t^j f(x)$, $x \in \Omega_1$, $t \in \mathbb{D}$).

Recall also that (see e.g. [4]) any bounded complete circular domain D in $\mathbb{C}^n$ has a schlicht envelope of holomorphy; what is more, its envelope of holomorphy $\hat{D}$ may be realized as a bounded complete circular domain in $\mathbb{C}^n$.

Example 4.3 (*See [3]*). Let Ω_1 be a bounded complete k-circular domain and Ω_2 a bounded balanced domain. Suppose that $f : \Omega_1 \to \Omega_2$ is a proper mapping such that $f^{-1}(0) = \{0\}$. Let $f = \sum_{j \geq p} f_j$, where f_j is k-homogeneous of order j. Assume that $f_p^{-1}(0) = 0$. Then $f = f_p$.

Proof. Repeating the argument used in Remark 4.2 we may assume that Ω_1 is a complete circular domain. Moreover, the mapping f may be extended to a proper holomorphic mapping between envelopes of holomorphy $\hat{f} : \hat{\Omega}_1 \to \hat{\Omega}_2$ such that $\hat{f}(\Omega_1) = \Omega_2$ and $\hat{f}^{-1}(\Omega_2) = \Omega_1$ (see e.g. [4, Theorem 2.12.5]). Therefore, we lose no generality assuming that Ω_1 and Ω_2 are pseudoconvex.

Then one may easily check that $A\|x\|^p \leq \|f_p(x)\| \leq B\|x\|^p$, $x \in \mathbb{C}^n$, for some positive A, B (use the fact that $f_p(x_1\|x\|^{-1}, \ldots, x_n\|x\|^{-1})$ is uniformly bounded for $x \neq 0$). This implies that $m\|x\|^p \leq \|f(x)\| \leq M\|x\|^p$, $x \in \Omega_1$, for some constants $m, M > 0$. Now it suffices to apply Proposition 4.1 to get that f is homogeneous. $\square$

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